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Interest Rates, Irreversibility, and Backward-Bending Investment

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This paper studies the effect of interest rates on investment in an environment where firms make irreversible investments with uncertain pay-offs. In this setting, changes in the interest rate affect both the cost of capital and the cost of delaying investment to acquire information. These two forces combine to generate an aggregate investment demand curve that is a backward-bending function of the interest rate. At low rates, increasing the interest rate raises investment by increasing the cost of delay.

How does an increase in interest rates affect capital investment by firms? The answer to this question has important implications for monetary and fiscal policies. The neoclassical theory of investment gives a simple answer: increasing the interest rate reduces investment by raising the cost of capital (Haavelmo, 1960; Jorgenson, 1963). This paper shows that the answer to this question is different when firms make irreversible investments with uncertain pay-offs. In this environment, investment is a *backward-bending* function of the interest rate.

To see the intuition, consider a pharmaceutical company deciding how quickly to proceed with investments in operations to produce new drugs. The firm is uncertain about drugs that will be successful and can acquire further information via R&D by delaying investment. The cost of delaying investment is that the firm cannot retire its outstanding debt as quickly, raising its interest expenses. Now consider how an increase in the interest rate will affect the firm's behaviour. A higher interest rate reduces the set of drugs that surpass the hurdle rate for investment, creating the standard cost of capital effect that acts to reduce the scale of investment. But a higher interest rate also makes the firm more eager to retire its debt quickly by investing immediately and earning profits sooner. This second "timing effect" acts to raise current investment. I show that these two opposing forces combine to generate a non-monotonic investment demand curve that is upward sloping at low interest rates.

I analyse a dynamic model where a continuum of profit-maximizing firms make binary investment decisions and can observe a noisy signal about the parameters that control pay-offs by postponing investment. The model builds on the large literature on irreversible investment and real options (*e.g.* Arrow, 1968; Bertola and Caballero, 1994; Dixit and Pindyck, 1994; Abel and Eberly, 1996). In the model analysed here, expected profits grow at a rate $g > 0$ when firms delay investment because the information acquired by delay reduces the probability of investing in an unsuccessful venture. Profits earned in subsequent periods are discounted at the interest rate, r. Therefore, firms invest immediately only if the expected profit from investment is positive and the expected growth in profits from delaying (g) is less than the interest rate. The backward-bending shape of the aggregate investment demand curve, $I(r)$, arises from this optimality condition. If r is low, g is likely to exceed r, compelling many firms to delay investment rather than investing

1. This paper is based on my undergraduate thesis at Harvard.

in period 1. On the other hand, when r is high, the expected return to investment is negative for many firms, deterring them from investing in period 1. Consequently, investment is maximized at an intermediate $r^* > 0$, and $I(r)$ is upward sloping from 0 to r^* and downward sloping above r^*.

A useful analogy in understanding this result is to interpret investment as the decision to cut a growing tree. The optimal time to cut a tree growing at a rate $g(t)$ that diminishes over time is when $g(t) = r$. When r is very low, trees that have already been planted are cut after a long time, reducing current investment. When r is very high, fewer trees are planted to begin with, because the return to investment is low, reducing the scale of investment. Hence, investment is low both when r is low and high. This logic results in a backward-bending $I(r)$ curve in a broad set of models where firms value the option to delay.[2]

The backward-bending property of the investment demand curve is robust to several generalizations of the basic model. First, the result holds when each firm chooses a scale of investment in each period. The aggregate economy in the extensive-margin model is isomorphic to a single firm making scale choices. Second, if firms have additional margins of choice beyond scale, such as choices about the composition of investment, the backward-bending shape that arises from learning effects is reinforced. For instance, if construction is cheaper when firms take a longer time to build (as in Alchian, 1959), they have an incentive to switch to slower building technologies when interest rates are low, reducing aggregate investment for reasons independent of learning.

The main result also holds in an equilibrium model of investment with competitive firms. To analyse the effects of competition, I extend the basic model to allow output prices and profit rates to be determined endogenously by market-clearing and free-entry conditions. In this environment, firms have a stronger incentive to invest early and beat the competition. However, if firms can earn sufficiently high quasi-rents (producer surplus) from investment in the short run, the equilibrium level of investment remains a backward-bending function of r. Intuitively, as long as the marginal firm values the option to delay in equilibrium—a condition that holds if identical competitors cannot enter and bid away all profits instantly—the interest rate continues to affect both the cost of delay and the cost of capital, thereby generating two opposing forces on investment demand in equilibrium.

The relationship between I and r derived here is of interest for two reasons. First, several studies have documented the importance of irreversibilities and the option to delay in firm-level investment behaviour (*e.g.* Caballero, Engel and Haltiwanger, 1995; Doms and Dunne, 1998; Caballero, 1999; Cooper and Haltiwanger, 2006). Analysing the relationship between interest rates and investment in such models is, therefore, important from the point of view of economic theory as well as macroeconomic policy. Second, the non-monotonic relationship is interesting from an empirical perspective, because several econometric studies have searched for a negative relationship between exogenous changes in the interest rate and aggregate investment demand without success (see Chirinko, 1993*a*,*b* for a review). This paper proposes a model that could explain the lack of a clear, monotonic relationship between I and r, at least in certain high-risk sectors of the economy where choices about timing of investment are important.

A natural question in this regard is whether the timing effects that generate the non-monotonic investment curve are empirically important. While empirical analysis is outside the scope of this paper, the learning structure of the model yields many predictions that could be tested in future work. For example, the model predicts that an increase in r is more likely to increase investment in sectors or times when the potential to learn is greater, that is, when signals

2. Capozza and Li (1994) and Jovanovic and Rousseau (2001, 2004) give related results on the effect of interest rates on land development and Initial Public Offerings (IPOs).

about future pay-offs are more informative and the variance of pay-offs is large. Examples that satisfy these conditions include start-ups or small businesses, especially in high-tech fields. The model also yields several additional testable predictions related to short-run vs. long-run changes in interest rates and investment and the effect of interest rates on observed profit rates.

The remainder of the paper is organized as follows. In the next section, I set up the basic firm-level model, solve for optimal investment behaviour, and aggregate the model to derive an investment demand curve. The main backward-bending investment result is given in Section 2. Section 3 generalizes the result to richer environments, including competitive equilibrium. Section 4 derives testable implications of the model. The final section offers concluding remarks. All proofs are given in the Appendix.

1. A MODEL OF INVESTMENT BY LEARNING FIRMS

I analyse a discrete-time learning model where firms making irreversible investment decisions maximize profits and are residual claimants in all states of the world. Since the analysis focuses on characterizing the shape of the investment *demand* curve, the interest rate is taken as exogenous throughout the paper.

I make two simplifying assumptions in the basic model, which are subsequently relaxed in Section 3. First, I assume that firms only decide whether to invest or not (the scale of investment is not flexible). Second, I ignore competitive forces by assuming that profit rates are fixed and unaffected by the behaviour of other firms in the economy. The basic model can be viewed as describing a firm that has a patent on an idea (*e.g.* a chemical compound) and is deciding whether to market its innovation (*e.g.* a new drug) by building a factory.

1.1. *Structure and assumptions*

Suppose a manager is deciding whether to invest in a new plant that can be built at cost C. The revenues from this investment are uncertain because demand for the firm's product is not known. There are two states of the world: the low-demand state ($\mu = 0$) and the high-demand state ($\mu = 1$).[3] Let R_μ denote total revenue from the project in state μ, and assume $R_1 > R_0$. To eliminate degenerate cases, assume that investment is unprofitable in the bad state $\forall r > 0$, that is, $R_0 < C$.

Investing in the plant allows the firm to start production in the next period, so revenue starts accruing one period after the investment is made. The decision to invest is irreversible—once the plant is built, it cannot be sold at any price and the firm does not make any further decisions.[4]

Let $\lambda_0 = P(\mu = 1)$ denote the manager's prior belief that the project will succeed. He can gain information about the state μ by delaying his investment decision and observing a signal z, for example, by conducting research. In the low-demand state, the signal z is drawn from a distribution $f(z)$; in the high-demand state, it is drawn from a distribution $g(z)$:

$$\mu = 0 \Rightarrow z \sim f(z) \quad \text{and} \quad \mu = 1 \Rightarrow z \sim g(z).$$

By postponing his decision, the manager can update his estimate of the probability of success to $\lambda_1 = P(\mu = 1 \mid z)$ after observing a realization of z and thereby make a more informed

3. The two-state assumption simplifies the exposition, but the results hold with a continuous state space.

4. Complete irreversibility is not essential. If there were a non-zero cost to undoing an investment, as in Abel and Eberly (1996), the firm would still be reluctant to commit resources to a venture of uncertain value. But if the investment decision were fully reversible and all money put in could be recovered, there would be no reason not to invest immediately, and the model would collapse to the neoclassical framework.

decision. The cost of this reduction in uncertainty is that a delayed investment yields revenues one period later, which have lower present value. I defer consideration of additional costs of delay, such as the cost of performing research necessary to obtain the signal or the loss of profits due to competition until Section 3.

The firm's investment opportunity is available for T periods. In the terminal period T, the firm must decide either to invest immediately or reject the project. In all periods $1 \leq t < T$, the firm chooses between investing immediately (i) or delaying its decision and learning (l). Let $\pi_t(\mu)$ denote the net pay-off in period 1 in dollars from investing in period t in state μ:

$$\pi_t(\mu) = \frac{1}{(1+r)^{t-1}} \left\{ \frac{R_\mu}{1+r} - C \right\}. \tag{1}$$

To simplify the discussion below, I focus on a two-period model ($T = 2$). However, all the results are proved in the Appendix for general T, including the limiting case of $T = \infty$.

1.2. *Optimal investment rule*

The optimal action in each period can be computed using backwards induction. To reduce notation, assume that the signal z is a scalar and that the likelihood ratio $\frac{g(z)}{f(z)}$ is monotonically and continuously increasing in z.[5] Let $V(i)$ denote the expected value of investing in period 1 and $V(l)$ the expected value of delay.

Lemma 1. *In period* 2, *the firm invests iff* $z > z^*$ *where* z^* *satisfies*

$$\frac{g(z^*)}{f(z^*)} = \frac{1-\lambda_0}{\lambda_0} \frac{C - R_0/(1+r)}{R_1/(1+r) - C}. \tag{2}$$

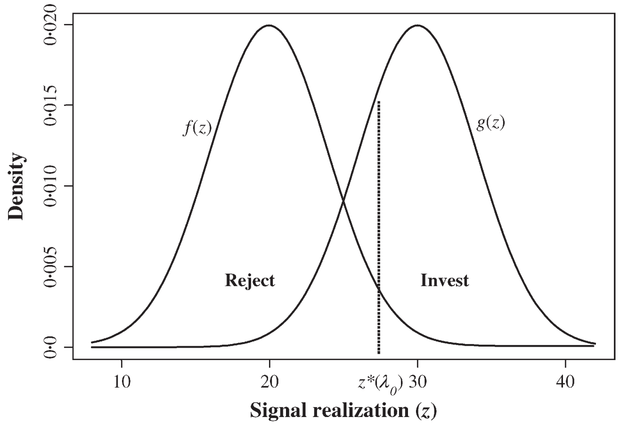
In period 1, *the firm invests iff*

$$\begin{aligned} V(i) &= \lambda_0 \left(\frac{R_1}{1+r} - C \right) + (1-\lambda_0) \left(\frac{R_0}{1+r} - C \right) \\ &> V(l) = \frac{1}{1+r} \left\{ \lambda_0 \beta(z^*) \left(\frac{R_1}{1+r} - C \right) + (1-\lambda_0) \alpha(z^*) \left(\frac{R_0}{1+r} - C \right) \right\}, \end{aligned} \tag{3}$$

where $\beta(z^*) \equiv \int_{z^*}^{\infty} g(z)dz$ *and* $\alpha(z^*) \equiv \int_{z^*}^{\infty} f(z)dz$.

The intuition for this result is as follows. In period 2, the firm chooses between investing or rejecting the project. The firm invests if the expected profit from investment is positive given the updated value of $P(\mu = 1)$ after observing signal z. The firm therefore invests in period 2 if the likelihood that the observed demand z came from the good distribution g is high, that is, if $\frac{g(z)}{f(z)}$ exceeds some threshold value. If $\frac{g(z)}{f(z)}$ is monotonic, there is a unique threshold z^* determined by the prior λ_0 and the profit–loss ratio such that investment is optimal iff $z > z^*$, as shown in Figure 1. The cut-off z^* is computed as in (2), so that the expected profit from investing in period 2 conditional on observing a signal $z = z^*$ that is 0. Intuitively, at the optimal threshold, the manager should be indifferent between investing and not investing in period 2. If he were not, there would either be a region of the signal state space where he is investing and earning negative expected profits or one where he is not investing and could have earned positive expected profits.

5. This monotonic likelihood ratio property holds for many distributions, including all one-parameter Natural Exponential Families.



Notes: Period 2 investment decision as a function of signal realization z. Optimal policy is a likelihood ratio test that results in a threshold rule: invest if $z > z^*(\lambda_0)$. Power of test ($\beta(z^*)$) is area under $g(z)$ distribution to the right of z^* and type 1 error rate ($\alpha(z^*)$) is corresponding area under $f(z)$ distribution.

FIGURE 1

Period 2 investment decision

Note that the firm's period 2 decision rule is formally equivalent to a likelihood ratio hypothesis test. The test has power $\beta(z^*)$, and type 1 error rate $\alpha(z^*)$. In the limiting case of noiseless signals, $\beta(x) = 1$ and $\alpha(x) = 0$ for all x. Under this decision rule, the firm invests in period 2 with probability β when $\mu = 1$ and probability α when $\mu = 0$.

In period 1, the firm chooses between investing or delaying and learning. The pay-off to investing is the expected profit in period 1, where the weight in the expectation is given by the prior belief, λ_0. The pay-off to learning, $V(l)$, is also a weighted average of profits in each state, but there are two changes in the formula. First, the relevant pay-off outcomes are π_2 instead of π_1—revenue is discounted more steeply, because it is earned one period later. Second, the weights in the profit expression are multiplied by the factors $\beta(z^*)$ and $\alpha(z^*)$. The term corresponding to the good state, $\pi_2(1)$, decreases by the weight $\beta(z^*) < 1$ because of the chance of rejecting the project when it is profitable. The test's benefit is that $\alpha(z^*) < 1$, placing less weight on the negative term corresponding to the bad state. In this model, the sole benefit of delaying investment is to reduce the probability of undertaking an unprofitable venture.

The period 1 investment rule is closely linked to the results of more general real options and optimal stopping models. To see this, let g denote the expected growth rate of profits by delaying, which is defined as the undiscounted expected profit in period 2 divided by the expected profit in period 1 (minus 1):

$$g = \frac{\{\lambda_0\beta(z^*)((R_1/(1+r)) - C) + (1-\lambda_0)\alpha(z^*)((R_0/(1+r)) - C)\}}{\lambda_0((R_1/(1+r)) - C) + (1+\lambda_0)((R_0(1+r)) - C)}. \tag{4}$$

Then we can rewrite the period 1 optimality condition for investment given in Lemma 1 as

$$V(i) > 0 \quad \text{and} \quad r > g. \tag{5}$$

This condition shows that it is optimal to invest immediately if (a) the expected profit from investment is positive and (b) the growth rate of profits from delaying, g, is smaller than the interest rate, r. If (b) is not satisfied, it is optimal to delay since doing so yields a higher expected rate of return than the interest cost. This condition mirrors standard results on optimal tree-cutting problems. It is optimal to cut a growing tree when the rate of return on the best alternative (r) exceeds the rate at which the tree grows (g). In the present model, the act of investment is equivalent to the act of cutting a tree, and the tree "grows" over time as firms acquire information and have higher expected profits.

The intuition embodied in (5) applies to a wide range of irreversible investment models (Dixit and Pindyck, 1994). Since the results below follow directly from this optimality condition, they hold in a general class of models and do not rely on the particular modelling details used here.

The two parts of equation (5) drive the two effects of interest rate changes on investment. The first part shows that a reduction in r makes individuals plant more trees (increasing the scale of investment), because more projects have positive expected value. The second part shows that a reduction in r also causes investors to cut trees later (postpone investment), because it is more likely that $g > r$. I show below that these two opposing forces combine to make investment a non-monotonic function of r.

1.3. *Aggregation*

To obtain a smooth aggregate investment demand curve, consider an economy populated by a continuum of learning firms with heterogeneous prior probabilities of success (λ_0's). Assume that the density of $\lambda_0, d\eta(\lambda_0)$ is continuous and places non-zero weight on all $\lambda_0 \in [0, 1]$.[6] Revenues from investment in each state and the learning technology are identical across firms. Assume for now that each firm's profit realization is independent of other firms' outcomes, so firms can ignore the behaviour of other firms when making investment decisions.

Under these assumptions, each firm follows Lemma 1 in making investment decisions. This allows us to characterize the decisions of all firms in the economy by a single threshold value λ_0^* that determines who invests in period 1 and who does not:

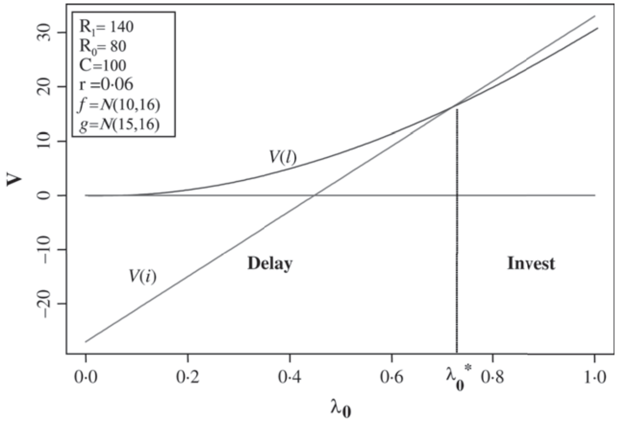
Lemma 2. *There is a unique λ_0^* at which the value of investing equals that of postponing. In period 1, firms with $\lambda_0 < \lambda_0^*$ delay their investment decision. Firms with $\lambda_0 \geq \lambda_0^*$ invest in period 1.*

Investment behaviour in the economy exhibits a simple pattern, as shown in Figure 2. Confident firms (λ_0 high) do not want to forego profits by delaying and invest immediately. The remaining firms, who are less certain about whether they have a profitable project, choose to wait and decide what to do in the next period based on the information they observe. The threshold λ_0^* thus determines the scale of investment in the economy.

It follows from Lemma 2 that aggregate period 1 investment is

$$I = \int_{\lambda_0^*}^{1} C d\eta(\lambda_0). \tag{6}$$

6. The assumption that all firms start their decision problem in period 1 is not restrictive, because the current belief λ_0 is a sufficient statistic for any previously acquired information. Firms that acquire information prior to period 1 simply have different values of λ_0.



Notes: Value functions and period 1 investment behaviour as a function of prior λ_0. $V(l)$ is value of delay and $V(i)$ is value of immediate investment. Firms with prior above threshold value λ_0^* invest in period 1.

FIGURE 2

Investment behaviour in the economy

2. INTEREST RATES AND INVESTMENT DEMAND

The following proposition characterizes the relationship between aggregate investment demand and the interest rate.

Proposition 1. *Investment demand is a backward-bending function of the interest rate.*

(i) $I(r=0)=0$ *and* $\lim_{r\to 0}\frac{\partial I}{\partial r}(r)=+\infty$
(ii) $r^*\equiv \arg\max_r I(r)>0$ *and* $r<r^*\Rightarrow \frac{\partial I}{\partial r}>0$ *and* $r>r^*\Rightarrow \frac{\partial I}{\partial r}\leq 0$.[7]

This proposition shows that $I(r)$ always has an upward-sloping segment from $r=0$ to $r=r^*>0$ followed by a downward-sloping segment thereafter. To see the intuition, first observe that if $r=0$, no one invests in the first period. Firms certain of success ($\lambda_0=1$) are indifferent between postponing and investing today, and all firms with lower priors strictly prefer delay (Lemma 2). Hence, $I(r=0)=0$: there is no reason to forego the free information one gets by waiting and learning if $r=0$. Increasing r from $r=0$ raises the cost of learning by delaying and *increases* aggregate investment by making the most confident firms invest immediately. At the other extreme, if $r>\frac{R_1}{C}-1$, projects are unprofitable in both states for all firms, and hence no one invests. Since no firms invest when r is low or high, it follows that $I(r)$ is non-monotonic.

To understand why the $I(r)$ curve is always *backward bending*, consider a firm with prior λ_0 such that investment in period 1 is optimal for some $r>0$.[8] Let us examine how the firm's pay-off to investment relative to delay, $V(i;\lambda_0)-V(l;\lambda_0)$, varies with respect to r. The expression $\frac{\partial\{V(i)-V(l)\}}{\partial r}$ can be decomposed into net present value (NPV) ($\frac{\partial V(i)}{\partial r}$) and learning ($-\frac{\partial V(l)}{\partial r}$)

7. More precisely, $\frac{\partial I}{\partial r}=0$ for $r>\frac{R_1}{C}-1$, the uninteresting case in which the interest rate is so high that investing is suboptimal even in the good state. Investment demand is strictly downward sloping ($\frac{\partial I}{\partial r}<0$) for all $r\in(r^*,\frac{R_1}{C}-1)$.

8. Such firms exist: for $\lambda_0=1$, $V(i;\lambda_0)>V(l;\lambda_0)\ \forall r>0\Rightarrow \exists\lambda_0'<1$ s.t. $V(i,\lambda_0')>V(l;\lambda_0')$ for some $r>0$ by continuity. Note that firms who never invest at any r do not affect the shape of $I(r)$.

effects:

$$\frac{\partial\{V(i)-V(l)\}}{\partial r}=\text{NPV}+L, \tag{7}$$

where

$$\text{NPV}=\frac{-1}{(1+r)^2}\{\lambda_0 R_1+(1-\lambda_0)R_0\}<0$$

$$L=\lambda_0\beta(z^*)\left\{\frac{2R_1}{(1+r)^3}-\frac{C}{(1+r)^2}\right\}+(1-\lambda_0)\alpha(z^*)\left\{\frac{2R_0}{(1+r)^3}-\frac{C}{(1+r)^2}\right\}>0. \tag{8}$$

The NPV effect makes an increase in r reduce the value of immediate investment, as in neoclassical investment models. The learning effect arises because the value of delaying is also affected by r. Via the L effect, a higher r reduces $V(l)$, creating a force that counteracts the conventional effect by making immediate investment more attractive.

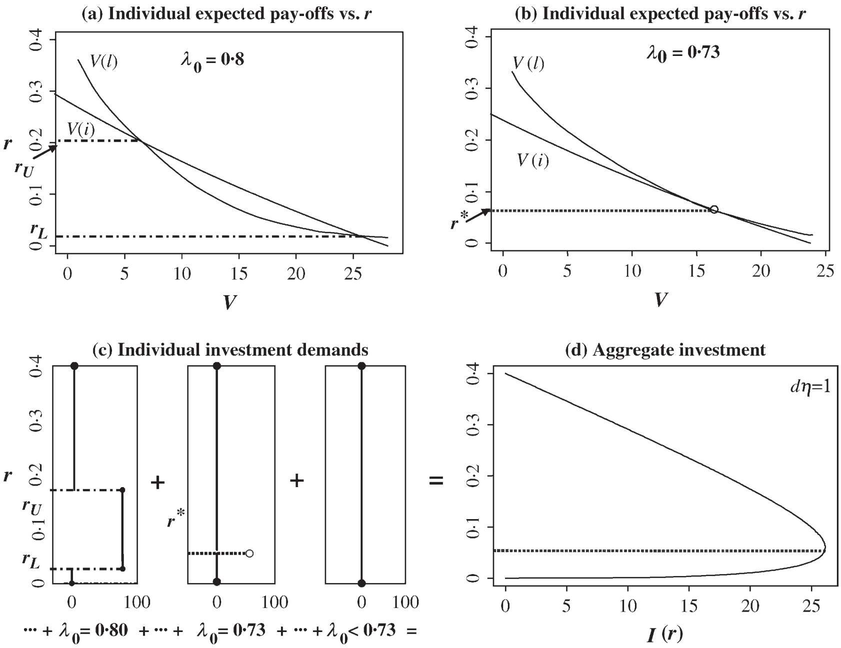
The magnitude of $L(r)$ diminishes relative to the magnitude of NPV(r) as r gets larger. Hence, for any given λ_0, there is exactly one value r at which NPV$(r)=-L(r)$. This implies that for a given firm, $V(i;\lambda_0)$ and $V(l;\lambda_0)$ intersect for at most two values of r, say $r_L(\lambda_0)$ and $r_U(\lambda_0)$. The firm-level investment demand curves thus all have the same form: invest iff $r_L(\lambda_0)\leq r\leq r_U(\lambda_0)$, as shown in Figure 3(a). The source of the non-monotonicity with respect to r is that a small increase in r causes $V(l)$ to fall *more than* $V(i)$ at $r_L(\lambda_0)$, increasing period 1 investment by firm λ_0. In contrast, an increase in r causes $V(l)$ to fall *less than* $V(i)$ at $r_U(\lambda_0)$, reducing the level of investment by the same firm.

The cut-off $r_L(\lambda_0)$ is decreasing in λ_0, while $r_U(\lambda_0)$ is increasing in λ_0. More confident firms have a larger interval of interest rates for which immediate investment is optimal. At the extremes, investors with $\lambda_0=1$ strictly prefer i for any $r\in(0,\frac{R_1}{C}-1)$, whereas investors with $\lambda_0=0$ prefer not to invest $\forall r>0$. There is exactly one value λ_0' such that $r_L(\lambda_0')=r_U(\lambda_0')$. For this firm, $V(i;\lambda_0)$ and $V(l;\lambda_0)$ are tangent at $r^*=r_L(\lambda_0')$, as shown in Figure 3(b). The λ_0' firm invests *only* if $r=r^*$.

Summing the individual non-monotonic step functions horizontally generates a smooth aggregate investment demand curve. Aggregate investment demand is a backward-bending function of r because the firm-level investment demand curves are non-monotonic step functions that are nested within each other as λ_0 falls, as shown in Figure 3(c). The slope of $I(r)$ approaches $+\infty$ as r tends to 0 because the most confident investors have little to gain by learning and immediately jump into the market when a small cost of delay is introduced. $I(r)$ is maximized at r^* because all firms who have $\lambda_0>\lambda_0'$ also invest at r^* by Lemma 2.

One concern with Proposition 1 is that the prediction that investment falls to 0 at low interest rates is empirically implausible. This prediction is an artefact of the stylized nature of the model. Two limitations of the model are important in this respect. First, in practice, some types of investment—such as replacement of depreciating machines—involve virtually no learning. This component of investment has a conventional downward-sloping relationship with r. In a model that allows for both non-learning and learning investment, total investment is positive at $r=0$. Nonetheless, total investment remains upward sloping at low r because $\lim_{r\to 0}\frac{\partial I}{\partial r}(r)=+\infty$ for the learning component. Second, even within the learning component, there are other non-interest costs to delay, such as research expenditures and loss of profits due to competition that are ignored in the model. I show in Section 3 that when these other costs are incorporated, $I(r=0)>0$, but $I(r)$ remains backward bending provided that these costs are not too large.

Note that in contrast with investment demand, the value of the firm always rises as r falls, because both $V(i)$ and $V(l)$ rise when r falls. Lower interest rates essentially lead to more investment in information rather than physical capital such as equipment and structures, ultimately yielding higher profit rates. If the measure of "investment" is broadened to include the value of



Notes: Firms compare $V(l)$ and $V(i)$ for each value of r (a, b), and compute their investment demands as functions of r (c). Summing these step functions horizontally yields $I(r)$ (d). Parameters used in simulation are the same as those in Figure 2.

FIGURE 3

The interest rate and period 1 investment

information, the conventional prediction that higher interest rates lower investment still holds. However, in so far as physical investment (as measured in balance sheets and national accounts) and information acquisition have different macroeconomic consequences, the non-monotonic effect of r on physical investment is of interest.[9]

3. EXTENSIONS

3.1. *Scale choice*

In the baseline model, each firm had a limited choice set: invest \$$C$ in either period 1 or 2. In practice, firms have some flexibility over their scale of investment in each period. To incorporate such scale choice, consider a firm that can set its level of investment in periods 1 and 2, I_1 and I_2, at any positive value. The restriction that investment must be positive captures irreversibility.[10] There are two states of the world ($\mu = 0, 1$), which differ in the price at which the output can be sold (p_μ). In state μ, an investment of I_1 in period 1 generates revenue of $p_\mu F(I_1)$ in period 2. An investment of I_2 in period 2 generates revenue of $p_\mu[F(I_1 + I_2) - F(I_1)]$ in period 3.

9. For example, if firms delay construction because r is low, building permits fall. In so far as building permits are viewed as an indicator of the economy's strength, this change in behaviour has relevance for economic policy.

10. In this model, downward adjustment of the capital stock has infinite cost, but upward adjustment (through additional investment) is costless. If upward adjustment is costly as well, the optimal investment rule differs, but $I_1(r)$ remains non-monotonic.

The marginal return to investment is diminishing: $F(I)$ is concave. To eliminate degeneracies, assume that $p_0 F'(0) < 1$, so that investment in the bad state is always unprofitable.

The information revelation structure of the model is the same as in Section 2: a signal z is observed at the end of period 1 and beliefs about μ are then updated. Let λ_0 denote the *ex ante* probability that $\mu = 1$. We can now generalize Proposition 1:

Proposition 2. *$I_1(r)$ is non-monotonic when firms choose scale:*

$$I_1(r=0)=0 \quad and \quad \exists r_1 > r_0 \quad s.t.\ I(r_1) > I(r_0).$$

The proof of this result parallels that for the extensive-margin case. When $r = 0$, total revenue is $p_\mu F(I_1 + I_2)$. Since there is no cost to delay in this case, there is no reason to invest immediately. Firms therefore set $I_1(r = 0) = 0$. Similarly, if r is sufficiently high, investment is unprofitable, and I_1 is again 0. Hence, investment demand is a non-monotonic function of r.

To understand why introducing scale choice does not change the main result, consider the following alternative model of scale choice. Suppose a firm has many projects in which it can invest, some of which have higher probabilities of success than others. The firm must make a binary decision about each project, but can choose the total number of projects to take up. As the firm raises investment, it is forced to choose projects with lower probabilities of success, making its profits a concave function of investment, as in the continuous scale-choice model. Since each project decision is made independently, investment decisions are determined exactly as in Lemma 2. Consequently, the total scale of investment by this firm, $I^{\mathrm{f}}(r)$, has the same form as equation (6), the expression for aggregate investment in the original model where several small firms make investment decisions on different projects. Firms are divisible, so total investment is identical if many small firms make decisions about one project each or one big firm makes investment decisions on several projects.

Since $I^{\mathrm{f}}(r)$ has the same form as (6), it follows that it also has the same backward-bending shape. This example shows that the original aggregate model with extensive-margin choices at the microeconomic level effectively contained a scale choice in the aggregate. In this sense, the basic model already contained the scale choice ("plant fewer trees") effect of increasing r. Modelling scale choice at the firm level instead of the aggregate level does not change the result.

3.2. *Investment composition decisions*

Firms can make many choices about projects beyond scale. For instance, they may choose technologies for construction or speed of delivery to market. To see how these "investment composition" decisions affect the shape of $I(r)$, suppose firms can choose between two construction methods, A and B. Method A requires the use of expensive building materials and is fast (*e.g.* one year to build). Method B involves less real investment but is slower (*e.g.* two years to build). At $r = 0$, time is costless, so the firm will use only method B. When r is very high, time is precious, and the firm will use only method A. For intermediate interest rates, the firm will use a combination of these two methods. Since method A involves more real investment than method B, the composition effect, holding scale fixed, makes $I(r)$ strictly upward sloping. As the scale effect dominates at high r—for sufficiently high r, it is best not to invest with any technology—$I(r)$ is downward sloping for high r when scale is endogenous. However, composition effects lengthen the upward-sloping segment of $I(r)$ and raise the investment-maximizing r^* generated by the basic model with only learning effects.

In the tree-cutting and planting analogy of Section 2, composition choices are the kinds of trees one plants (oak or apple). Increases in the interest rate have three effects in this environment:

(1) plant fewer trees; (2) cut trees later; and (3) plant trees that mature later. Effects 2 and 3 act to make $I(r)$ upward sloping at low interest rates. Generalizing the model to allow composition choices thus reinforces the main result and illustrates that learning is not the only reason that firms may reduce investment in response to an interest rate cut.

3.3. *Competitive equilibrium*

The analysis thus far has assumed that investors enjoy rents from investments and do not face competitive pressures. While patent and copyright protection limit competition in some cases, in practice, most firms face some competition in the long run that bids away rents.

To model competition among learning firms, consider a market with a continuum of firms that have different product concepts (*e.g.* different pain relievers). Each firm makes a binary decision to invest in a plant that costs C. Investment in period t yields revenues in period $t+1$. A firm that invests in period t ends up with either a good product that sells at price p_t in period $t+1$ or a bad product that is worthless (sells for 0). Firms' outcomes are independent: the probability that any single firm succeeds is unrelated to the other firms' behaviour and outcomes.

Each firm enters period 1 with a prior probability of having a good product of λ_0. As in the basic model, there is a smooth distribution of λ_0's to capture heterogeneous expectations. Firms that delay investment receive independent signals about whether their products are good at the end of period 1, which are used to update beliefs. In the second period, firms choose between investing immediately or rejecting the project (delaying again is not possible).

The price p_t is determined endogenously to equate cumulative supply with demand in equilibrium. Let $I_t \in [0,1]$ denote aggregate investment in period t, defined as the measure of firms that invest in period t. Let I_t^{c} denote cumulative investment up to and including period t. The inverse-demand function for good products sold for the first time in period t is given by an arbitrary downward-sloping function $p(I_t^{\mathrm{c}})$.[11] This implies that the price of the product falls over time ($p_1 > p_2$). To eliminate degenerate cases, assume that $\frac{p(I_2^{\mathrm{c}}=0)}{1+r} > C$ and $\frac{p(I_2^{\mathrm{c}}=1)}{1+r} < C$, so that there exist firms on the margin of investing in period 2.

To model free entry in the long run, assume that profits are bid to 0 after the first period in which a particular product is sold. After this point, other firms can replicate the technology, forcing the original firm to sell at marginal cost. A firm that invests in period 1 thus has a chance to earn money in period 2 only; firms that invest in period 2 can earn money in period 3 only. The one-period lag captures adjustment costs, which prevent competitors from bidding away inframarginal quasi-rents (short-run surplus) by selling an identical product instantly.[12] The pharmaceutical industry is a concrete example of this type of competitive structure: first-movers can earn large profits in the short run (*e.g.* Aspirin), while subsequent firms with slightly different products can also earn temporary rents (Tylenol, Advil) until their profits are also bid away by generics who replicate the original products.

In this setting, the expected profit from immediate investment (i) and learning by delaying (l) for a firm with prior λ_0 in period 1 is

$$V(i,\lambda_0) = \frac{\lambda_0 p_1}{1+r} - C$$
$$V(l,\lambda_0) = \frac{1}{1+r}\left\{\lambda_0\beta(z^*)\left(\frac{p_2}{1+r} - C\right) + (1-\lambda_0)\alpha(z^*)(-C)\right\}.$$

11. This is equivalent to assuming that prices depend on the supply of good products instead of total investment because the supply of good products is a monotonic function of I_t^{c}.

12. This model of competition parallels neoclassical competitive production theory, where producer surplus is positive in the short run and falls to 0 in the long run.

An equilibrium is defined by two conditions: (1) markets clear in each period and (2) all firms optimize—those with $V(i, \lambda_0) > V(l, \lambda_0)$ at the market price vector (p_1, p_2) invest in period 1 and those who delay and have positive expected profits in period 2 invest in period 2. The following lemma establishes existence and uniqueness of equilibrium in this model and characterizes investment behaviour in equilibrium.

Lemma 3. *In period* 1 *equilibrium, there is a unique price vector* (p_1, p_2) *and threshold* λ_0^* *at which*

$$V(i, \lambda_0^*) = V(l, \lambda_0^*) > 0.$$

Firms with $\lambda_0 < \lambda_0^*$ *delay their investment decision.*

Firms with $\lambda_0 \geq \lambda_0^*$ *invest in period* 1.

The key point of Lemma 3 is that the marginal investor in period 1 equilibrium earns *strictly positive* expected profits from immediate investment. Unlike in the neoclassical model of competition, profits are not driven to 0 at the margin in the period 1 equilibrium. To understand this result, first consider period 2 decisions. Since there is no further option to delay, a firm invests in period 2 if its expected return to investment at the market-clearing price exceeds the cost of investment. Consequently, there is a threshold value λ_1^* such that only firms with updated probabilities of success $\lambda_1 > \lambda_1^*$ invest in period 2. The marginal firm with belief λ_1^* earns zero profits in equilibrium. But the infra-marginal firms who have higher λ_1's earn positive profits in expectation. These firms are able to earn temporary quasi-rents despite being in a competitive market because they have a better technology (such as a better chemical compound or human capital) that cannot be instantly replicated by other firms.

Now turn to period 1 behaviour. There is some probability that the marginal investor in period 1 will be one of the infra-marginal investors in period 2. Hence the value of postponing must be strictly positive for this indifferent firm. The reason that expected profits are not driven to 0 at the margin in the period 1 equilibrium is again heterogeneity in success probabilities. Other firms are free to enter the market and try to capture the positive rents, but they have lower probabilities of success than the indifferent firm and therefore can earn higher expected profits by delaying.

Since the option value of delaying is positive for the marginal period 1 investor in equilibrium, changes in r continue to affect that firm's behaviour via both an NPV and learning effect, as in the basic model. The existence of these two opposing forces suggests that period 1 investment demand, $I_1(r)$, may be non-monotonic in competitive equilibrium. This result cannot be established using the same proof as in the basic model because there is now a non-interest cost to waiting, so $I_1(r = 0) > 0$.[13] Even at a zero interest rate, the most confident (highest λ_0) firms will invest immediately to take advantage of the high initial price they can extract. Nonetheless, one can obtain a simple condition under which the investment demand curve in this model is upward sloping at low r.

Proposition 3. *Let* λ_0^* *and* λ_1^* *denote the success probabilities of the marginal (indifferent) investors in periods* 1 *and* 2, *respectively. Then* $\partial I_1/\partial r(r = 0) > 0$ *if at* $r = 0$,

$$\lambda_0^* \beta(\lambda_0^*) > \lambda_1^*. \tag{9}$$

13. However, at very high r, it remains the case that investment is suboptimal for all firms, so aggregate investment falls to 0 as $r \to \infty$. Hence, $I(r)$ must have a downward-sloping segment in the competitive model.

This condition requires that the marginal investor in period 1 have a significantly higher success probability than the marginal investor in period 2. Since the marginal investor in period 2 earns zero profits in equilibrium, this condition guarantees that the marginal investor in period 1 can gain substantial rents by delaying and investing in period 2, since he is likely to be an infra-marginal investor in that period.

To understand why $I(r)$ is non-monotonic when (9) is satisfied, consider two extreme examples. First, suppose signals are perfect, so that $\beta(\lambda_0^*) = 1$. In this case, the distribution of λ_1 is a degenerate two-point distribution, and if supply is sufficiently large, price is driven down to $p_2 = C$ for those who invest in the second period. Since firms cannot earn any profits if they delay investment, the option to delay is worthless. The model collapses into the conventional single-period model, where r has only a conventional cost-of-capital (scale) effect and $I(r)$ is strictly downward sloping. Correspondingly (9) does not hold in this case because $\lambda_1^* = 1$. This example illustrates that the "timing effect" of r can emerge only if delaying is a real option that has value in equilibrium. Condition (9) essentially guarantees that the option to delay has value.

Now consider a second example, where signals are imperfect. Suppose the demand curve for the good product is

$$p_t = h(I_t^c) + K$$

where K is a constant and $\partial h/\partial I_t^c < 0$ so that demand is downward sloping. Suppose the cost of investment is

$$C = C_0 + \frac{1}{2}K.$$

In this example, K controls the variance of pay-offs: high K yields higher profits in the good state, but a bigger loss in the bad state. The following result establishes that (9) holds when pay-off uncertainty is sufficiently high, implying that $I(r)$ is upward sloping at low r:

Corollary to Proposition 3. *For K sufficiently large,* $\partial I/\partial r(r = 0) > 0$.

The mechanics underlying this result are straightforward. As K becomes large, the threshold for investment in period 2 approaches $\lambda_1^* = \frac{1}{2}$ because firms earn approximately the same amount in the good state $(\frac{K}{2})$ as they lose in the bad state. In period 1, increased uncertainty makes delay more attractive for each firm, raising the threshold for investment λ_0^*. Therefore, as the amount of uncertainty grows larger, λ_0^* and consequently $\beta(\lambda_0^*)$ approach 1 while λ_1^* approaches $\frac{1}{2}$, so that (9) is eventually satisfied.

Intuitively, in a very risky environment, the incentive to delay and acquire information is large; so only the most confident investors take advantage of high equilibrium prices in period 1. However, in period 2, when there is no further opportunity to learn, many lower-capability firms are willing to take risky but positive NPV risks. This creates large infra-marginal rents in the second period for the marginal period 1 investor, who is confident of success. These large rents become less valuable when interest rates rise, compelling the marginal firm to start investing immediately when r rises from $r = 0$, and raising aggregate investment in competitive equilibrium.

The model of competition analysed here is specialized, but the qualitative results can be extended to richer settings where entry dynamics are endogenous and prices fall gradually as competitors enter the market. The general point is that if the option to delay is sufficiently valuable for the marginal investor in equilibrium (taking into account the potential loss of profits from competitive forces, research costs, and other costs of delay), the investment demand curve is upward sloping at low r.

4. TESTABLE PREDICTIONS

This section presents a set of comparative statics that could be used to test the empirical relevance of the model in future work. These predictions are derived in the basic model of Section 2 for simplicity.

4.1. *Potential to learn*

An increase in r is most likely to increase investment in environments with a high potential to learn. A formal definition of the "potential to learn" is necessary to make this conjecture precise. Intuitively, a firm can learn more rapidly if signal noise is lower, that is, if it easier to distinguish whether z is drawn from f or g. Recall that any firm's second period decision is the outcome of a hypothesis test. I will say that "signal noise" rises if the power of the test, $\beta(x) = \int_x^\infty g(z)dz$, falls while the type 1 error rate, $\alpha(x) = \int_x^\infty f(z)dz$, rises for *all* cut-off values x below the point at which f and g are indistinguishable. Formally, let $s(f, g)$ denote the level of signal noise with densities f and g, and x' the unique point at which $\frac{g(x')}{g(x')} = 1$. Then

$$s(f_1, g_1) > s(f_2, g_2) \quad \text{if} \quad \beta_2(x) > \beta_1(x) \quad \text{and} \quad \alpha_1(x) > \alpha_2(x)\ \forall x < \min(x_1', x_2'), \tag{10}$$

where x_j' is s.t. $\frac{g(x_j')}{f(x_j')} = 1$. An example of an increase in signal noise according to this definition is a rightward shift of $g(z)$ or a leftward shift of $f(z)$ in Figure 1. Note that this definition is an incomplete ordering since it does not rank all distributions in terms of signal noise.

Before turning to the relationship between signal noise and $\frac{\partial I}{\partial r}$, it is useful to first establish the connection between signal noise and the level of I itself.

Lemma 4. *An increase in signal noise increases investment*

$$s(f_1, g_1) > s(f_2, g_2) \Rightarrow I(f_1, g_1) > I(f_2, g_2).$$

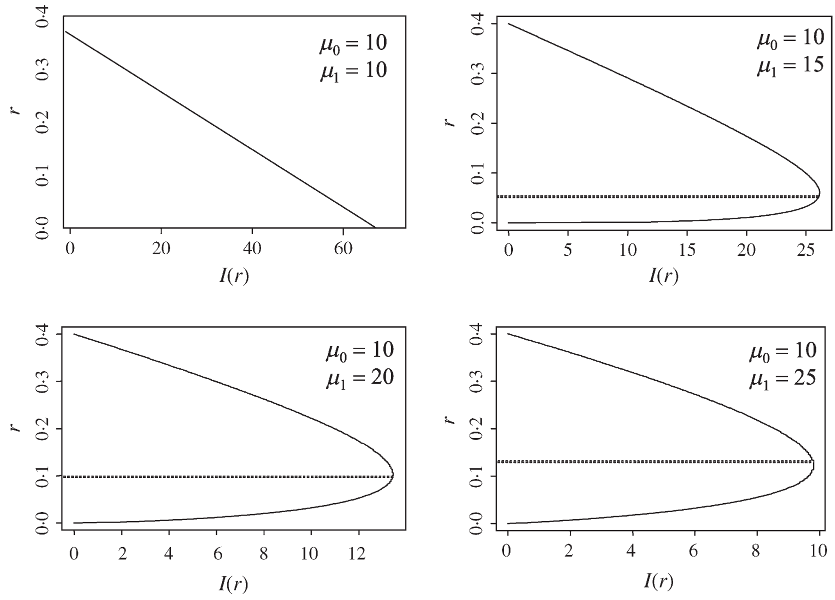
When signal noise rises, a firm's ability to learn about the true value of μ by waiting is reduced. This reduces the value of delaying investment, making aggregate investment rise. Cukierman (1980) gives an analogous result: increases in the variance of earnings reduce investment by raising the value of delay.

How does an increase in signal noise affect the shape of $I(r)$? To build intuition, consider the extreme case of totally uninformative signals ($f = g$). In this case, the model collapses into the neoclassical model, and the $I(r)$ curve is downward sloping, that is, $r^* = 0$. This observation suggests that the potential to learn should be positively associated with r^*; that is, the upward-sloping segment of the investment–demand curve should be larger in industries or times where there is more to be learned. The following proposition establishes that this is indeed the case provided that the pay-off in the bad state is sufficiently low or equivalently, the variance of returns is sufficiently high relative to the expected return.

Proposition 4. *There exists $\overline{R_0} > 0$ s.t. if $R_0 < \overline{R_0}$, a reduction in signal noise raises r^*:*

$$s(f_2, g_2) < s(f_1, g_1) \Rightarrow r_2^* > r_1^*.$$

Figure 4 illustrates this result by showing $I(r)$ for distributions with progressively lower signal noise. To see the intuition for the result, observe that changes in signal noise affect only



Notes: This figure shows $I(r)$ for four pairs of signal distributions f and g. The distributions are normal with a mean of μ_0 for f and μ_1 for g and a standard deviation of 16. Other parameters are the same as those in Figure 2.

FIGURE 4

Signal noise and $I(r)$

$V(l)$, leaving $V(i)$ unaffected for each firm. An increase in r is more likely to raise an aggregate investment if it tends to reduce $V(l)$ more than $V(i)$, making immediate investment preferable. When signal uncertainty is lowered, $V(l)$ changes in two ways. First, firms have a higher probability of investing in the good state in period 2 (β rises). Second, firms have a lower probability of investing in the bad state (α falls). The first effect makes expected period 2 profits more sensitive to the interest rate, since there is a higher probability of earning revenues in the good state. The second effect goes in the opposite direction, since there is a *lower* probability of earning revenues in the bad state.

If R_0 is small, the second effect is small in magnitude relative to the first, and so $V(l)$ is more sensitive to r overall. For instance when $R_0 = 0$, an increase in r has no effect at all on revenues in the bad state. Therefore, provided that R_0 is low, an increase in r is more likely to reduce $V(l)$ relative to $V(i)$ for each firm and thereby raise aggregate investment when signal uncertainty is lower. The low R_0 condition on the result requires that the variance of earnings be high relative to the mean profit rate, which is essentially a requirement that information about the state of demand is valuable.

4.2. *Short run vs. long run*

I now turn to the effects of changes in r on total investment over a longer horizon, taking into account changes in investment behaviour beyond the current period. For this analysis, it is necessary to consider the T period formulation of the model instead of the two-period special case discussed above. In this model, the firm has the option to delay investment in every period from 1 to $T-1$.[14]

In the T-period model, total investment from period 1 to t is given by

$$I_{1,t} = \sum_{s=1}^{t} I_s = \int_{\lambda_0^*}^{1} C d\eta(\lambda_0) + \sum_{n=2}^{t} \int_{0}^{\lambda_0^*} P_1(I_s \mid \lambda_0) C d\eta(\lambda_0) \tag{11}$$

where $P_1(I_s \mid \lambda_0)$ is the probability that a firm with prior λ_0 ends up investing in period s. The next proposition analyses the relationship between $I_{1,t}$ and r.

Proposition 5.

(i) $I_{1,t}(r)$ *is a backward-bending function of* r $\forall t < T$:

$$r_{1,t}^* \equiv \arg\max_r I_{1,t}(r) > 0 \quad \textit{and} \quad r < r_{1,t}^* \Rightarrow \frac{\partial I_{1,t}}{\partial r} > 0 \quad \textit{and} \quad r > r_{1,t}^* \Rightarrow \frac{\partial I_{1,t}}{\partial r} < 0$$

(ii) *The upward-sloping portion of the* $I_{1,t}(r)$ *curve becomes smaller as* t *rises:*

$$r_{1,t}^* > r_{1,t+1}^*.$$

The first part of the proposition is driven by the same two effects that make the response of investment demand in period 1 to a change in r non-monotonic. If $r = 0$, all the firms will postpone their decision until T and $I_{1,t}(r = 0) = 0$. Similarly, if r is large, $I_{1,t} = 0$. Hence, investment over the first t periods is maximized at intermediate interest rates.

To understand the second result, observe that the growth in profits from delay diminishes over time because the marginal return to information falls as more knowledge is accumulated. When r falls, investors may delay investment for a few periods to acquire information, but eventually acquire enough information that further delay is undesirable. Since reductions in r generate temporary delays in investment, the conventional cost of capital effect starts to dominate at lower levels of r in the long run. Consequently, the investment-maximizing $r_{1,t}^*$ falls with t.[15]

Proposition 5 implies that the long-run elasticity of investment demand is more negative than the short-run elasticity of investment demand when firms learn over time. If the near-zero existing estimates of the short-run interest elasticity of investment demand are due to learning effects, interest rate reductions from policies that stimulate savings could, nonetheless, increase investment over a longer horizon.

4.3. *Average profit rates*

In the neoclassical model, a higher interest rate increases the average rate of return of investments that are undertaken by driving out low NPV ventures. This result is also modified when firms learn over time.

To analyse the average rate of return, we must identify the level of *ex post* profitable ($\mu = 1$) and *ex post* unprofitable ($\mu = 0$) investment by specifying how frequently a project that a manager expects to succeed with probability λ_0 actually does succeed. A natural benchmark is rational

14. See Lemma 1A in the Appendix for a characterization of optimal investment behaviour in the T-period model.

15. Unlike learning effects, which die away in the long run as firms acquire perfect information, composition effects may never subside. Hence, when composition effects are permitted (as in Section 3.2), the long-run investment demand curve can have a substantial upward-sloping segment.

expectations: $P_{\lambda_0}[\mu = 1] = \lambda_0$. In this case, the average (net) profit rate among investments that are undertaken in period 1 is given by:

$$\rho(r) = \frac{\int_{\lambda_0^*}^{1} \lambda_0(R_1 - C) + 1(1-\lambda_0)(R_0 - C) d\eta(\lambda_0)}{\int_{\lambda_0^*}^{1} C d\eta(\lambda_0)}. \tag{12}$$

Proposition 6. *The average profit rate ρ is a backward-bending function of the interest rate:*

$$r < r^* \Rightarrow \frac{\partial \rho}{\partial r} > 0 \quad \text{and} \quad r > r^* \Rightarrow \frac{\partial \rho}{\partial r} < 0.$$

As established in Proposition 1, when $r < r^*$, an increase in the interest rate draws the marginal investor with prior $\lambda_0^*(r)$ into the period 1 pool of investors. This firm has the lowest probability of success among the set of firms who are investing. Consequently, it pulls down the average rate of return in the overall pool. Conversely, when $r > r^*$, an increase in r eliminates the marginal investor with prior $\lambda_0^*(r)$, who has the lowest probability of success in the pool of investors, increasing the average rate of return. The average observed profit rate on current investment is thus a backward-bending function of r. Building on earlier results, an increase in the interest rate is more likely to lower the average observed rate of return when the potential to learn is greater and in the short run relative to the long run.

4.4. *Temporary interest rate and tax changes*

I now discuss a few comparative statics for temporary interest rate changes. Formal statements are omitted since these results are simple extensions of the preceding propositions.

First, consider the effect of an *unanticipated* temporary increase in the interest rate. Let r_{1t} denote the per-period interest rate between periods 1 and $t < T$. An unanticipated increase in r_{1t} (holding the interest rate fixed in all other periods) is more likely to reduce current investment than a permanent increase in r because one can take advantage of lower future costs of capital by delaying investment. If the potential to learn is sufficiently high, $I(r_{1t})$ is backward bending, with a smaller upward-sloping segment than $I(r)$. When the potential to learn is low, $I(r_{1t})$ is strictly downward sloping. The longer the duration of an interest rate change, the less the incentive to postpone investment following a temporary increase, and the larger the range of parameters over which $I(r_{1t})$ is upward sloping.

Now consider the effect of a temporary *anticipated* change in the interest rate that begins in period $s > 1$ and lasts until period $t > s$. An anticipated increase in r_{st} is more likely to raise current investment than a permanent change in r because one can take advantage of lower current costs of capital by investing immediately. In fact, if the change is anticipated sufficiently far in advance, current investment may be a strictly upward-sloping function of r_{st}. In contrast, future investment falls when r_{st} rises because of the inter-temporal substitution.

Together, these results indicate that the shape of the yield curve on bonds—which embodies investors' expectations of future interest rates—should have a significant effect on current and future investment. When the yield curve becomes steeper, current investment should rise relative to subsequent investment.

A final set of predictions relates to the effect of tax changes. In the neoclassical model, taxes matter only through the user cost of capital. In irreversible investment models, both the user cost and the discount rate matter, and different tax policies may affect these two quantities differently. For example, accelerated depreciation provisions change the user cost but not the discount rate, since there is no additional incentive to delay from accelerated depreciation itself.

Hence, accelerated depreciation should unambiguously raise investment. In contrast, changes in capital income taxation affect equilibrium interest rates, thereby changing the discount rate and user cost simultaneously, with potentially non-monotonic effects on investment.

5. CONCLUSION

This paper has explored the effect of interest rates on investment in an environment where firms making irreversible investments learn over time. The main result is that at low rates, an increase in r *increases* investment demand by enlarging the set of projects for which r exceeds the return to delay.

The empirical relevance of the model for explaining how interest rate changes affect aggregate investment depends on the extent to which firms re-time investments in response to changes in market conditions. There is some evidence that firms delay investment when faced with increased uncertainty, as predicted by irreversible investment models (*e.g.* Leahy and Whited, 1996; Bulan, Mayer and Somerville, 2004; Bloom, Bond and Reenen, 2006). There is also some evidence that cross-sectional variation in interest rates are related to the speed of real estate development (Capozza and Li, 2001), and time-series fluctuations in interest rates affect the timing of IPO decisions (Jovanovic and Rousseau, 2004). In future work, it would be interesting to test whether exogenous interest rate changes affect the timing and profitability of irreversible investments in high-risk industries.

APPENDIX

All proofs for the baseline case (Section 2) are given in a model with an arbitrary decision horizon T. The corresponding results discussed in the text are for the $T = 2$ case, unless otherwise noted. We begin by restating Lemma 1 (the optimal investment rule when $T = 2$) for arbitrary T in Lemma 1A.

Lemma 1A (Optimal Investment Rule in the *T* period model). *Let λ_{t-1} denote the firm's prior in period t. In period T, the firm invests iff $z_{T-1} > z^*_{T-1}$ where z^*_{T-1} satisfies*

$$\frac{g(z^*_T)}{f(z^*_T)} = \frac{1-\lambda_{T-1}}{\lambda_{T-1}} \frac{-\pi_T(0)}{\pi_T(1)}. \tag{1}$$

In any period $t < T$, the firm invests iff $V_t(i) > V_t(l)$:

$$V_t(i) = \lambda_{t-1}\pi_t(1) + (1-\lambda_{t-1})\pi_t(0) \tag{2}$$

$$V_t(l) = \sum_{s=t+1}^{T} \lambda_{t-1} P_t(I_s \mid \mu = 1)\pi_s(1) + (1-\lambda_{t-1}) P_t(I_s \mid \mu = 0)\pi_s(0) \tag{3}$$

where

$$\frac{\lambda_t}{1-\lambda_t} = \frac{\lambda_0}{1-\lambda_0} \frac{g(z_1,\ldots,z_t)}{f(z_1,\ldots,z_t)} = \frac{\lambda_0}{1-\lambda_0} \frac{g(z_1)\cdots g(z_t)}{f(z_1)\cdots f(z_t)} \tag{4}$$

and $\forall t > 0$: $z^*_t(z_1,\ldots,z_{t-1})$ is uniquely defined by

$$V_t(i, z^*_t, \lambda_{t-1}) = V_t(l, z^*_t, \lambda_{t-1})$$

and

$$P_t(I_{t+1} \mid \mu = 1) = \int_{z^*_t}^{\infty} g(z_t)dz_t$$

$$P_t(I_{t+1} \mid \mu = 0) = \int_{z^*_t}^{\infty} f(z_t)dz_t$$

and $\forall s \in \{t+2, \ldots, T\}$:

$$P_t(I_s \mid \mu = 1) = \int_{-\infty}^{z_t^*} \int_{-\infty}^{z_{t+1}^*} \cdots \int_{-\infty}^{z_{s-2}^*} \int_{z_{s-1}^*}^{\infty} g(z_{s-1})dzg(z_{s-2})dz \cdots g(z_{t+1})dzg(z_t)dz$$

$$P_t(I_s \mid \mu = 0) = \int_{-\infty}^{z_t^*} \int_{-\infty}^{z_{t+1}^*} \cdots \int_{-\infty}^{z_{s-2}^*} \int_{z_{s-1}^*}^{\infty} f(z_{s-1})dzf(z_{s-2})dz \cdots f(z_{t+1})dzf(z_t)dz.$$

Proof of Lemma 1A. *Period T.* Bayes rule and the assumption that $z_t \perp z_s$ for $t \neq s$ directly imply equation (4). In period T, the pay-off to investing $V_T(i)$ is computed using the updated belief $\lambda_T(z)$ about the probability with which $\mu = 1$ occurs. The firm invests iff $V_T(i) > 0 \Rightarrow \frac{\lambda_T(z)}{1-\lambda_T(z)} > \frac{-\pi_T(0)}{\pi_T(1)}$. This results in the period T decision rule in (1) using (4) and the assumption that $\frac{g(z)}{f(z)}$ is monotonically increasing.

Period $T-1$. The remainder of the proof is done by backward induction starting with period $T-1$, where the firm is faced with a two-period decision problem. First, note that $V_{T-1}(i)$ is computed simply by taking an expectation over the π_{T-1} function. To compute $V_{T-1}(l)$, integrate the expected pay-off in period T over the prior density of z:

$$V_{T-1}(l) = \int_{-\infty}^{\infty} \max(V_T(d_1), V_T(i))dm(z) = \int_{-\infty}^{z^*} V_T(d_1) + \int_{z^*}^{\infty} V_T(i)dm(z) \tag{5}$$

where $dm(z) = \lambda_0 g(z) + (1-\lambda_0) f(z)$ is the unconditional density on z.

$$\Rightarrow V_{T-1}(l) = \lambda_0 \pi_T(1) \int_{z_{T_D-1}^*}^{\infty} g(z)dz + (1-\lambda_0)\pi_T(0) \int_{z_{T_D-1}^*}^{\infty} f(z)dz. \tag{6}$$

Next, I establish that the firm follows a threshold rule for investment in period $T-1$: $\exists$ unique $z_{T-2}^*(z_1, \ldots, z_{T-3})$ defined by $V_{T-1}(i, z_{T-2}^*, \lambda_{T-3}) = V_{T-1}(l, z_{T-2}^*, \lambda_{T-3})$ s.t. that investing is optimal iff $z_{T-2} > z_{T-2}^*$. It is sufficient to show that $\exists$ unique λ_{T-2}^* s.t. $V_{T-1}(i, \lambda_{T-2}^*) = V_{T-1}(l, \lambda_{T-2}^*)$ and that $\lambda_{T-2} > \lambda_{T-2}^*$ makes investing optimal. To see that there is a unique λ_{T-2}, rewrite:

$$V_{T-1}(i) = \lambda_{T-2} b + (1-\lambda_{T-2})a$$

$$V_{T-1}(l) = \lambda_{T-2} b' + (1-\lambda_{T-2})a',$$

where $a = \pi_{T-1}(0)$, $a' = P_{T-1}(I_T \mid \mu = 0)\pi_T(0)$, $b = \pi_{T-1}(1)$, and $b' = P_{T-1}(I_T \mid \mu = 1)\pi_T(1)$.

Note that $V_{T-1}(l, \lambda_{T-1} = 0) = 0 > V_{T-1}(i, \lambda_{T-1} = 0)$. By assumption, $\exists \lambda_{T-1}$ s.t. $V_{T-1}(l, \lambda_{T-1}) \leq V_{T-1}(i, \lambda_{T-1})$

By the Intermediate Value Theorem, $\exists \lambda_{T-2}^*$ s.t. $V_{T-1}(i, \lambda_{T-2}^*) = V_{T-1}(l, \lambda_{T-2}^*)$.

Since $\frac{\partial V_{T-1}(l)}{\partial \lambda_{T-2}} > 0$, it follows that $V_{T-1}(l, \lambda_{T-2}^*) > 0$. Now observe that

$$\left.\frac{\partial\{V_{T-1}(i) - V_{T-1}(l)\}}{\partial \lambda_{T-2}}\right|_{\lambda_{T-2}^*} = b - a + a' - b > 0$$

because $V_s(i, \lambda_{T-2}^*) = V_s(l, \lambda_{T-2}^*) \Leftrightarrow \lambda_{T-2}^* b + (1-\lambda_{T-2}^*)a = \lambda_{T-2}^* b' + (1-\lambda_{T-2}^*)a' > 0$.

Therefore, at any λ_{T-2}^*, we must have

$$\left.\frac{\partial\{V_{T-1}(i) - V_{T-1}(l)\}}{\partial \lambda_{T-2}}\right|_{\lambda_{T-2}^*} > 0,$$

which implies that λ_{T-2}^* is unique. Hence $V_{T-1}(i, \lambda_{T-2}) > V_{T-1}(l, \lambda_{T-2})$ iff $\lambda_{T-2} > \lambda_{T-2}^*$.

Induction. Finally, I show that when $V_{t+1}(\cdot)$ has the form in (2) and (3), $V_t(\cdot)$ also has the same form. $V_t(i)$ is easily computed. To compute $V_t(l)$, recall that the firm invests in period $t+1$ iff $z_t > z_t^*$, where z_t^* has already been computed.

Therefore

$$V_t(l) = \int_{z_t^*}^{\infty} [\lambda_t \pi_{t+1}(1) + (1-\lambda_t)\pi_{t+1}(0)][\lambda_{t-1} g(z_t) + (1-\lambda_{t-1}) f(z_t)] dz_t$$

$$+ \int_{-\infty}^{z_t^*} V_{t+1}(l, \lambda_t)[\lambda_{t-1} g(z_t) + (1-\lambda_{t-1}) f(z_t)] dz_t$$

$$= \sum_{s=t+1}^{T} \lambda_{t-1} P_t(I_s \mid \mu = 1)\pi_s(1) + (1-\lambda_{t-1}) P_t(I_s \mid \mu = 0)\pi_s(0). \tag{7}$$

Finally, arguments analogous to those given above establish that $\exists$ unique z_{t-1}^* s.t.

$$V_{t-1}(i, z_{t-1}^*, \lambda_{t-2}) = V_s(l, z_{t-1}^*, \lambda_{t-2})$$

and that if $z_{t-1} > z_{t-1}^*$ the investor will invest in period t. This completes the induction. $\parallel$

Proof of Lemma 2. The proof follows directly from the second step of Lemma 1A. In any period t, $\exists$ unique λ_{t-1}^* s.t. $V_t(i, \lambda_{t-1}^*) = V_t(l, \lambda_{t-1}^*)$ and that $\lambda_{t-1} > \lambda_{t-1}^* \Leftrightarrow$ Invest. Applying this to $t = 1$ gives the result. $\parallel$

Proof of Lemma 3. In period 2, a firm invests if $\frac{\lambda_1 p_2}{1+r} > C$. Since $\frac{\partial p_2}{\partial I_2^c} < 0$, it follows that there is a unique cut-off value λ_1^* and a corresponding price $p_2(I_2^c(\lambda_1^*))$ such that

$$\frac{\lambda_1^* p_2(I_2^c(\lambda_1^*))}{1+r} = C. \tag{8}$$

In equilibrium, all firms with $\lambda_1 > \lambda_1^*$ invest, all other firms stay out, and the market clears at the resulting price p_2 by construction. Therefore, for any firm with given prior λ_0, there is a cut-off value $z^*(\lambda_0)$ such that the firm invests in period 1 iff $z > z^*$.

Now consider period 1 investment behaviour. Firms invest in period 1 if

$$V(i, \lambda_0) = \frac{\lambda_0 p_1}{1+r} - C \tag{9}$$

$$> V(l, \lambda_0) = \lambda_0 \beta(z^*(\lambda_0)) \left[\frac{p_2}{(1+r)^2} - \frac{C}{1+r} \right] + (1-\lambda_0)\alpha(z^*(\lambda_0)) \left[-\frac{C}{1+r} \right] \tag{10}$$

where α and β are defined as in the text. Note that $V(l, \lambda_0)$ is strictly positive in equilibrium if $\lambda_0 > 0$. It follows from Lemma 2 that for a given price vector (p_1, p_2), there is a unique λ_0^* such that firms with $\lambda_0 > \lambda_0^*$ invest in period 1 and the remainder delay. Since $\partial p_t / \partial I_t^c < 0$, $\partial \lambda_0^* / \partial p_1 < 0$, and $\partial \lambda_0^* / \partial p_2 > 0$, there is a unique price vector (p_1, p_2) at which all firms are optimizing and markets clear in both periods. Thus, equilibrium investment is characterized by a price vector (p_1, p_2) and a cut-off value λ_0^* such that

$$V(i, \lambda_0^*, p_1) = V(l, \lambda_0^*, p_1, p_2) > 0. \qquad \parallel \tag{11}$$

Proof of Lemma 4. Take any (f_1, g_1), (f_2, g_2) s.t. $s(f_1, g_1) > s(f_2, g_2)$. For the firm with $\lambda_0' = \lambda_0^*(f_1, g_1)$,

$$V_1(l) = \sum_{s=2}^{T} \lambda_0 P_1(I_s \mid \mu = 1)\pi_s(1) + (1-\lambda_0) P_1(I_s \mid \mu = 0)\pi_s(0) \tag{12}$$

$$V(i, \lambda_0') = \lambda_0' \left\{ \frac{R_1}{1+r} - C \right\} + (1-\lambda_0') \left\{ \frac{R_0}{1+r} - C \right\}. \tag{13}$$

The shift from (f_1, g_1) to (f_2, g_2) only affects $V(l, \lambda_0')$. It follows from the definition of a reduction in signal noise that for any s, $P_1(I_s \mid \mu = 1)$ is higher under (f_2, g_2) than under (f_1, g_1), while $P_1(I_s \mid \mu = 0)$ is lower. Hence,

$$V(l, \lambda_0'; f_2, g_2) > V(l, \lambda_0'; f_1, g_1) = V(i, \lambda_0'). \tag{14}$$

Lemma 2 implies that $\lambda_0^*(f_2, g_2) > \lambda_0' = \lambda_0^*(f_1, g_1)$. Since $\frac{\partial I}{\partial \lambda_0^*} < 0$, the result follows. ‖

Proof of Proposition 1.

(i) Equations (2) and (3) imply that in period 1

$$V(i) = \lambda_0 \pi_1(1) + (1-\lambda_0)\pi_1(0) \tag{15}$$

$$V(l) = \sum_{s=2}^{T} \lambda_0 P_1(I_s \mid \mu = 1)\pi_s(1) + (1-\lambda_0) P_1(I_s \mid \mu = 0)\pi_s(0) \tag{16}$$

where $\pi_t(\mu) = \frac{R_\mu}{(1+r)^t} - \frac{C}{(1+r)^{t-1}}$. Now suppose $r = 0$. Then

$$V(i) = \lambda_0\{R_1 - C\} + (1-\lambda_0)(R_0 - C)$$

$$V_1(l, \lambda_0) = \lambda_0\{R_1 - C\} \sum_{t=2}^{T} P_1(I_t \mid \mu = 1) + (1-\lambda_0)\{R_0 - C\} \sum_{t=2}^{T} P_1(I_t \mid \mu = 0).$$

For $\lambda_0 = 1$, the definitions in Lemma 2 imply that $z_t^* = \infty$ for $t < T$ and $z_T^* = -\infty \Rightarrow \sum_{t=2}^{T} P_1(I_t \mid \mu = 1) = 1$. Therefore $\lambda_0 = 1 \Rightarrow V(i, \lambda_0) = (\pi_1 - \delta) = V(l, \lambda_0)$. Hence $\lambda_0^*(r = 0) = 1 \Rightarrow I(r = 0) = 0$.
To establish that $\lim_{r \to 0} \frac{\partial I}{\partial r}(r) = +\infty$, note first that $\frac{\partial I}{\partial r} = \frac{\partial I}{\partial \lambda_0^*} \times \frac{\partial \lambda_0^*}{\partial r}$.
By the Implicit Function Theorem,

$$\frac{\partial \lambda_0^*}{\partial r} = -\left.\frac{\partial V(i,\lambda_0)/\partial r - \partial V(l,\lambda_0)/\partial r}{\partial V(i,\lambda_0)/\partial \lambda_0 - \partial V(l,\lambda_0)/\partial \lambda_0}\right|_{\lambda_0^*}. \tag{17}$$

Let $\frac{\partial \lambda_0^*}{\partial r} = \frac{N(r)}{D(r)}$. It can be shown that $N(r = 0) = C - R_1 < 0$ and $D(r) > 0\ \forall r > 0$.
Since $D(r = 0) = 0$ and $\frac{\partial I}{\partial \lambda_0^*} = -\lambda_0^* d\eta(\lambda_0^*) < 0$ it follows that $\lim_{r \to 0} \frac{\partial I}{\partial r}(r) = +\infty$.

(ii) Note that $C < R_1 \Rightarrow \exists r'$ s.t. $\frac{R_1}{1+r'} = C$. Then

$$r \geq r' \Rightarrow V(i, \lambda_0) < 0\ \forall \lambda_0 \leq 1 \Rightarrow \lambda_0^*(r) = 1 \Rightarrow I(r) = 0.$$

Given that (i) $I(0) = I(r') = 0$, (ii) $I(r)$ is continuous, and (iii) $[0, r']$ is compact, it follows that $I(r)$ has an interior maximum $r^* \in (0, r')$.
To prove uniqueness of r^*, recall

$$\frac{\partial I}{\partial r} = \frac{\partial I}{\partial \lambda_0^*} \times \frac{\partial \lambda_0^*}{\partial r} = \frac{\partial I}{\partial \lambda_0^*} \frac{N(r)}{D(r)}. \tag{18}$$

Since I is continuous and $\frac{\partial V(i,\lambda_0) - \partial V(l,\lambda_0)}{\partial \lambda_0}(\lambda_0^*) > 0\ \forall r > 0$, $N(r) = 0$ at any critical r. A sufficient condition for r^* to be unique is that $N(r)$ has a unique root. I establish this by showing that $N(r) = 0 \Rightarrow \frac{\partial N}{\partial r}(r) > 0$.
Given that $\frac{\partial V(l) - \partial V(i)}{\partial r}(r^*) = 0$ and $V(l, \lambda_0^*(r^*)) = V(i, \lambda_0^*(r^*))$, it follows that

$$\frac{\partial N}{\partial r}(r^*, \lambda_0^*(r^*)) = \frac{1}{1+r^*} \partial \left[\frac{(1+r)(\partial V(l) - \partial V(i))}{\partial r} \right] \Big/ \partial r = \frac{1}{1+r^*} \partial [M(r)]/\partial r \tag{19}$$

where

$$\begin{aligned} M(r) &= \frac{\partial((1+r)V(l))(-\partial((1+r)V(i))}{\partial r} \\ &= -\lambda_0 \beta(z^*) \frac{R_1}{(1+r)^2} - (1-\lambda_0)\alpha(z^*) \frac{R_0}{(1+r)^2} + C, \end{aligned} \tag{20}$$

and

$$\beta \equiv \sum_{t=2}^{T} \frac{P_1(I_t \mid \mu = 1)}{(1+r)^{t-1}}, \alpha \equiv \sum_{t=2}^{T} \frac{P_1(I_t \mid \mu = 0)}{(1+r)^{t-1}}.$$

Differentiating (20) gives

$$\partial[M(r)]/\partial r = \lambda_0\beta(z^*)\frac{2R_1}{(1+r)^3} + (1-\lambda_0)\alpha(z^*)\frac{2R_0}{(1+r)^3} - \frac{\partial\beta}{\partial r}\frac{R_1}{(1+r)^2} - \frac{\partial\alpha}{\partial r}\frac{R_0}{(1+r)^2}. \tag{21}$$

To sign (21), note that $\frac{\partial\alpha}{\partial r}(r^*) < 0$ and $\frac{\partial\beta}{\partial r}(r^*) < 0$. This is easiest to see when $T = 2$, where

$$\frac{\partial\beta}{\partial r} = -\frac{\partial z^*}{\partial r}g(z^*),\ \frac{\partial\alpha}{\partial r} = -\frac{\partial z^*}{\partial r}f(z^*).$$

Note that $\frac{\partial z^*}{\partial r} > 0$ because $\frac{g(z^*)}{f(z^*)} = \frac{1-\lambda_0}{\lambda_0}\frac{C-R_0/(1+r)}{R_1/(1+r)-C}$ and $\partial[\frac{g(z)}{f(z)}]/\partial z > 0$. Therefore $\frac{\partial\alpha}{\partial r} < 0$ and $\frac{\partial\beta}{\partial r} < 0$. When $T > 2$, the final step can be established using $\frac{\partial P_1(I_t|\lambda_0)}{\partial r}(r^*) < 0\ \forall t > 1$, which follows from $\frac{\partial^2\{V(l,\lambda_0)-V(i,\lambda_0)\}}{\partial\lambda_0\partial r} < 0$ (see proof of Proposition 4 below) and the definition of $P_1(I_t|\lambda_0)$. Since $\frac{\partial\alpha}{\partial r} < 0$ and $\frac{\partial\beta}{\partial r} < 0$, it follows that $\partial M(r)/\partial r(r^*) > 0 \Rightarrow \partial N(r)/\partial r(r^*) > 0$. ||

Proof of Proposition 2. Define $\overline{p} = \lambda_0 p_1 + (1-\lambda_0)p_0$ as the expected price *ex ante* and note that $\overline{p} = \int_{z=-\infty}^{\infty}\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\}dm(z)$ where $dm(z)$ denotes the marginal distribution of z, as defined in the text.

In period 2, after observing z, the firm updates its beliefs to $\lambda_1(z)$ and chooses $I_2(z)$ to

$$\max\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\}\frac{F(I_1+I_2)-F(I_1)}{(1+r)^2} - \frac{I_2(z)}{1+r}\text{ s.t. } I_2 \geq 0. \tag{22}$$

Let $I_2^*(z)$ denote the optimal level of period 2 investment as a function of z. At an interior optimum, optimal investment $I_2^*(z)$ satisfies:

$$F'(I_1+I_2^*(z))\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\} = 1+r. \tag{23}$$

When $F'(I_1)\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\} < 1+r$, $I_2^*(z) = 0$.

The value function in period 2 is

$$V_2(I_1) = \int_{z=-\infty}^{\infty}\left[\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\}\frac{F(I_1+I_2^*(z))-F(I_1)}{(1+r)^2} - \frac{I_2^*(z)}{1+r}\right]dm(z). \tag{24}$$

The value function in period 1 is:

$$\begin{aligned} V_1 &= \overline{p}\frac{F(I_1)}{1+r} - I_1 + V_2(I_1) \\ &= \int_{z=-\infty}^{\infty}[\lambda_1(z)p_1 + (1-\lambda_1(z))p_0]\left[\frac{F(I_1)}{1+r} + \frac{F(I_1+I_2(z))-F(I_1)}{(1+r)^2}\right] - \frac{I_2(z)}{1+r} - I_1 dm(z). \end{aligned} \tag{25}$$

Given the optimality condition for $I_2^*(z)$, it follows that

$$\frac{\partial V_1(r=0)}{\partial I_1} = \int_{z=-\infty}^{\infty}[\lambda_1(z)p_1 + (1-\lambda_1(z))p_0][F'(I_1+I_2^*(z))-1]dm(z) < 0$$

since for z sufficiently small, $F'(I_1)\{\lambda_1(z)p_1 + (1-\lambda_1(z))p_0\} < 1$. Hence $V_1(r=0)$ is maximized when $I_1(r=0) = 0$.

It is easy to show that as $r \to \infty$, $I_1 \to 0$ and that $\exists r' > 0$ s.t. $I_1(r') > 0$. These observations, coupled with continuity of $I_1(r)$, establish the result. ||

Proof of Proposition 3. Observe that $\frac{\partial I_1}{\partial r} = \frac{dI_1}{d\lambda_0^*}\frac{d\lambda_0^*}{dr}$. The implicit function theorem implies

$$\frac{d\lambda_0^*}{dr} = -\frac{dV(i,\lambda_0)/dr - dV(l,\lambda_0)/dr}{dV(i,\lambda_0)/d\lambda_0 - dV(l,\lambda_0)/d\lambda_0} = -\frac{N}{D}. \tag{26}$$

At $r = 0$, the denominator of this expression is

$$D = p_1 - \beta(\lambda_0^*)p_2 + (\beta(\lambda_0^*) - \alpha(\lambda_0^*))C.$$

Since $\beta > \alpha$ and $p_1 > p_2$, it follows that $D > 0$. The numerator is

$$N = -\frac{\lambda_0 p_1}{(1+r)^2} + \frac{\lambda_0 dp_1/dr}{1+r} - \left\{\lambda_0\beta\left(-2\frac{p_2}{(1+r)^3} + \frac{dp_2/dr}{(1+r)^2} + \frac{C}{(1+r)^2}\right) + (1-\lambda_0)\alpha\left(\frac{C}{(1+r)^2}\right)\right\}. \tag{27}$$

Using the fact that $V(i,\lambda_0^*) = V(l,\lambda_0^*)$, N simplifies at $r=0$ to

$$N(r=0)|_{p\text{-fixed}} = \lambda_0^*(r=0)\beta(\lambda_0^*)p_2 - C$$

if dp_t/dr is 0.

I now show that $N(r=0)|_{p\text{-fixed}} \Longrightarrow \frac{\partial I}{\partial r}(r=0) > 0$ using a proof by contradiction. Suppose that $\frac{\partial I_1}{\partial r}(r=0) < 0$. In this case, $dp_1/dr > dp_2/dr > 0$ because $\partial I_1^{\mathrm{c}}/\partial I_1 = 1$ and $\partial I_2^{\mathrm{c}}/\partial I_1 < 1$ and $\partial p_t/\partial I_t^{\mathrm{c}} < 0$. It follows that $N(r=0) > 0$ when p is variable. Since $N > 0$ and $D > 0$, $\frac{d\lambda_0^*}{dr} < 0$, which implies $\frac{\partial I_1}{\partial r}(r=0) > 0$. But this contradicts the supposition. Therefore $\frac{\partial I_1}{\partial r}(r=0) > 0$ if $N(r=0)|_{\text{p-fixed}} > 0$.

Since $\lambda_1^* p_2 = C(1+r)$ in period 2 equilibrium, $N(r=0)|_{\text{p-fixed}} > 0$ if $\frac{\lambda_0^*\beta(\lambda_0^*)}{\lambda_1^*} > 1$. Hence

$$\frac{\lambda_0^*\beta(\lambda_0^*)}{\lambda_1^*} > 1 \Longrightarrow \frac{\partial I_1}{\partial r}(r=0) > 0. \qquad \|$$

Corollary to Proposition 3. At $r=0$, the period 2 threshold for investment is characterized by the following condition:

$$\lambda_1^* = \frac{C_0 + \frac{1}{2}K}{h(I_2^{\mathrm{c}}) + K}.$$

Since I_2^{c} is bounded, it follows that

$$\lim_{K\to\infty} \lambda_1^* = \frac{1}{2}.$$

I now establish that raising K lowers period 1 investment using a proof by contradiction. Consider the behaviour of the marginal period 1 firm, with $\lambda_0 = \lambda_0^*$. At the original equilibrium, the marginal investor has

$$V(i,\lambda_0) = \lambda_0\left(h(I_1^{\mathrm{c}}) + \frac{1}{2}K - C_0\right) + (1-\lambda_0)\left[-\left(C_0 + \frac{1}{2}K\right)\right] \tag{28}$$

$$= V(l,\lambda_0) = \lambda_0\beta(z^*(\lambda_0))\left[(h(I_2^{\mathrm{c}}) + K) - \left(C_0 + \frac{1}{2}K\right)\right] + (1-\lambda_0)\alpha(z^*(\lambda_0))\left[-\left(C_0 + \frac{1}{2}K\right)\right]. \tag{29}$$

Suppose that increasing K raises I_1. This would require that $dV(i,\lambda_0)/dK > dV(l,\lambda_0)/dK$. Recall that $\partial I_1^{\mathrm{c}}/\partial I_1 = 1$ and $\partial I_2^{\mathrm{c}}/\partial I_1 < 1$. Hence, $\partial h/\partial I_t^{\mathrm{c}} < 0$ implies $\partial V(i,\lambda_0)/\partial I_1 < \partial V(l,\lambda_0)/\partial I_1$. In addition, observe that $\partial V(i,\lambda_0)/\partial K < \partial V(l,\lambda_0)/\partial K$ because $\beta > \alpha$. Consequently, both the direct effect of increasing K and the indirect effect of equilibrium price changes lower $V(i,\lambda_0)$ relative to $V(l,\lambda_0)$ for the marginal investor. This makes the marginal firm drop out of the period 1 investment pool, lowering I_1. But this contradicts the supposition. Therefore increasing K must strictly *lower* I_1. Hence, as $K\to\infty$, $\lambda_0^*\to 1$, while implies $\beta(\lambda_0^*)\to 1$. Meanwhile $\lambda_1^*\to\frac{1}{2}$. It follows that (9) must hold as $K\to\infty$, as claimed. $\|$

Proof of Proposition 4. Take any (f_1,g_1) and (f_2,g_2) s.t. $s(f_2,g_2) < s(f_1,g_1)$. Let r_1^* and r_2^* denote the investment-maximizing values of r in each case, and let $\lambda_0' = \lambda_0^*(r=r_1^*,\ f_1,g_1)$ and $\lambda_0'' = \lambda_0^*(r=r_1^*,\ f_2,g_2)$. By Lemma 4, $\lambda_0'' > \lambda_0'$.

It can be shown that

$$\frac{\partial^2\{V(l,\lambda_0) - V(i,\lambda_0)\}}{\partial\lambda_0\partial r} = \partial\left\{\left(\frac{R_1}{(1+r)^2} - \frac{C}{1+r}\right)\beta(z^*) - \left(\frac{R_0}{(1+r)^2} - \frac{C}{1+r}\right)\alpha(z^*)\right.$$
$$\left. - \left(\left(\frac{R_1}{1+r} - C\right) - \left(\frac{R_0}{1+r} - C\right)\right)\right\}/\partial r < 0.$$

Since $\frac{\partial V(l,\lambda_0';f_1,g_1)}{\partial r} = \frac{\partial V(i,\lambda_0';f_1,g_1)}{\partial r}$ at $r = r_1^*$, it follows that $\frac{\partial V(l,\lambda_0'';f_1,g_1)}{\partial r} < \frac{\partial V(i,\lambda_0'';f_1,g_1)}{\partial r}$ at $r = r_1^*$.

Using logic similar to that in Lemma 4, $s(f_2, g_2) < s(f_1, g_1) \Rightarrow \beta_2 > \beta_1$ and $\alpha_2 < \alpha_1$. Observe that

$$\frac{\partial V(l)}{\partial r} = -\lambda_0 \beta \left\{ \frac{2R_1}{(1+r)^3} - \frac{C}{(1+r)^2} \right\} - (1-\lambda_0)\alpha \left\{ \frac{2R_0}{(1+r)^3} - \frac{C}{(1+r)^2} \right\}.$$

If $R_0 < C/2$, both terms become more negative when β rises and α falls; hence

$$\frac{\partial V(l, \lambda_0''; f_2, g_2)}{\partial r} < \frac{\partial V(l, \lambda_0''; f_1, g_1)}{\partial r} < \frac{\partial V(i, \lambda_0''; f_1, g_1)}{\partial r} = \frac{\partial V(i, \lambda_0''; f_2, g_2)}{\partial r}$$

$$\Rightarrow \frac{\partial I}{\partial r}(r = r_2^*; f_1, g_1) = \frac{\partial I}{\partial \lambda_0^*} \times \left. \frac{\partial V(l,\lambda_0)/\partial r - \partial V(i,\lambda_0)/\partial r}{\partial V(i,\lambda_0)/\partial \lambda_0 - \partial V(l, \lambda_0)/\partial \lambda_0} \right|_{\lambda_0''} > 0.$$

Therefore $R_0 < C/2 \Rightarrow r_1^* > r_2^*$ by Proposition 1. ‖

Proof of Proposition 5.

(i) For any $t < T$, $r = 0 \Rightarrow \lambda_0^* = 1$ and $z_1^* = z_2^* = \ldots = z_t^* = \infty$ by Lemma 1A.
Therefore $I_{1,t}(r = 0) = 0$. As above, $I_{1,t}(r) = 0$ for $r > r' > 0$ for r' s.t. $\frac{R_1}{1+r'} = C$.
By the Intermediate Value Theorem, $\exists r_{1,t}^*$ that is a critical point and global maximum for $I_{1,t}$.
Uniqueness of $r_{1,t}^*$ follows from an argument equivalent to that given in Proposition 1 to show that $r < r_{1,t}^* \Leftrightarrow \frac{\partial I_{1,t}}{\partial r} > 0$.

(ii) I will show that, $\frac{\partial I_{1,t}}{\partial r}(r) < 0 \Rightarrow \frac{\partial I_{1,t+1}}{\partial r}(r) < 0$, which establishes the result. Observe that

$$I_{1,t}(r) = \sum_{s=1}^{t} I_s(r) = \int_{\lambda_0^*}^{1} C \, d\eta(\lambda_0) + \sum_{s=2}^{t} \int_{0}^{\lambda_0^*} P_1(I_s \mid \lambda_0) C \, d\eta(\lambda_0). \tag{30}$$

Therefore

$$\frac{\partial I_{1,t}}{\partial r} = -\frac{\partial \lambda_0^*}{\partial r} C d\eta(\lambda_0^*) \left\{ 1 - \sum_{s=2}^{t} \frac{1}{(1+r)^{s-1}} P_1(I_s \mid \lambda_0^*) \right\} + C \int_{0}^{\lambda_0^*} \left[\sum_{s=2}^{t} \frac{\partial P_1(I_s \mid \lambda_0)}{\partial r} \right] d\eta(\lambda_0). \tag{31}$$

Now suppose $r > r^*$, the investment-maximizing interest rate for period 1. Then $\frac{\partial \lambda_0^*}{\partial r} > 0$ implies the first term is negative in the expression above. To evaluate the sign of the second term, note that $\frac{\partial P_1(I_t|\lambda_0)}{\partial r} < 0 \; \forall \lambda_0 < \lambda_0^*$ when $r > r^*$. This follows from $\frac{\partial^2 \{V(l,\lambda_0) - V(i,\lambda_0)\}}{\partial \lambda_0 \partial r} < 0$ and the definition of $P_1(I_t \mid \lambda_0)$. Since $P_1(I_t \mid \lambda_0) > 0 \; \forall t > 1$ and $\frac{\partial P_1(I_t|\lambda_0)}{\partial r} < 0 \; \forall t > 1$, it follows by inspection that $\frac{\partial I_{1,t}}{\partial r}(r) < 0 \Rightarrow \frac{\partial I_{1,t+1}}{\partial r}(r) < 0$. ‖

Proof of Proposition 6. Define $\rho(r) = \int_{\lambda_0^*}^{1} \lambda_0(R_1 - C) + (1-\lambda_0)(R_0 - C)) d\eta(\lambda_0) / \int_{\lambda_0^*}^{1} C d\eta(\lambda_0)$.
Note that

$$\partial \rho / \partial r = \left(-\frac{\partial \lambda_0^*}{\partial r} \right) \frac{C d\eta(\lambda_0^*)}{[\int_{\lambda_0^*}^{1} C d\eta(\lambda_0)]^2} A, \tag{32}$$

where

$$A = \int_{\lambda_0^*}^{1} [\lambda_0^*(R_1 - C) + (1-\lambda_0^*)(R_0 - C)] - [\lambda_0(R_1 - C) + (1-\lambda_0)(R_0 - C)] d\eta(\lambda_0). \tag{33}$$

Since $\int_{\lambda_0^*}^{1} (\lambda_0^* - \lambda_0) \, d\eta(\lambda_0) < 0$, it follows that $A < 0$. Hence $r < r^* \Rightarrow \frac{\partial \lambda_0^*}{\partial r} < 0 \Rightarrow \frac{\partial \rho}{\partial r} < 0$ and $r > r^* \Rightarrow \frac{\partial \lambda_0^*}{\partial r} > 0 \Rightarrow \frac{\partial \rho}{\partial r} > 0$ ‖

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